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Fractional Fuzzy Optimal Control Problem Governed by Fuzzy System

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ABSTRACT

In this article, we found the solution of a fractional fuzzy controlled system which has fuzzy initial conditions by using α-cuts and representing fuzzy numbers as complex numbers. Next, a fractional fuzzy optimal control problem (FFOCPs) has been considered to optimize a fuzzy controlled objective function. We use fractional Pontryagin Maximum Principle to solve the fuzzy optimal control problem in fractional order. Then, in the last section, we have some examples to apply this method showing the effect of solving fuzzy optimal controlled system in fractional order.

Key words: *Fractional system, Fuzzy control, Fractional Pontryagin Maximum Principle, Fractional derivative, Fractional integral.*

INTRODUCTION

Optimal control theory is an oriented subject for obtaining optimal strategies to control dynamical systems and many other events that they need to be control to be useful. Naturally, many real events and dynamical systems have uncertainty in their input, out put and manner ; in this regard, we know that fuzziness is a way to express an uncertain phenomena in real world. So, by importing fuzziness in the optimal control theory, problems can be displayed better with control parameters in real world as physical models and dynamical systems. While, in the last years, fractional calculus performs very important roles in mathematics, mechanics, and other subject. Many dynamical systems and events have much better performance when they modeled by using fractional differential equations.

If a fuzzy differential equation in fractional order contains a control variable, then we have a fractional fuzzy optimal control problem (FFOCPs). Zhu [1] applied Bellman's optimal principle to make the principle of optimality for fuzzy optimal control ; and Diamond and Kloeden [2] discussed on existence the solution of such control systems. Then, Park et.al [3] obtained the sufficient conditions for fuzzy control systems. Filev and Angelove [4] had solved fuzzy optimal control of nonlinear system with fuzzy mathematical programing. Z. Qin [5] solved the time-homogeneous fuzzy optimal control problems, discounted objective function. In [6] by considering the generalized differentiability authors, used new solutions for fuzzy two point boundary value problems for Hukuhara differentiability. Georgiou et.al [7] discussed nth-order fuzzy differential equations with initial value conditions. Nieto et.al [8] found numerical methods for solving fuzzy differential equations. Agrawal [9], used the Lagrange multipliers technique, to obtain necessary conditions for optimality of fuzzy optimal control problems. In this paper, we consider a fractional fuzzy control problem and turn it to two fractional control problem and use fractional Pontryagin Maximum Principle to solve it.

Consider the following fractional fuzzy control problem :

Min: $\int_a^b f_0(t, \tilde{x}(t), \tilde{u}(t))dt$ (1)

S. to :

$$\begin{cases} \left(D_{a+}^{\beta}\tilde{x}\right)(t) = f\big(t, \tilde{x}(t), \tilde{u}(t)\big); \\ \qquad \tilde{x}(a) = \tilde{x}_0 \,, \end{cases}$$

where $t \in (a, b) \subseteq \mathcal{R}$, $\tilde{x}$ is a fuzzy variable, $\tilde{x}_0$ is a fuzzy initial condition, $\tilde{u}$ is the fuzzy control variable, f and f_0 are two given functions respect to t, $\tilde{x}$ and $\tilde{u}$; here $\left(D_{a+}^{\beta}\tilde{x}\right)(t)$ denotes the left Riemann-Liouville derivative at order $\beta \in (0,1)$. The problem (1), that is fuzziness respect to x and u and also governed by a differential equation with fractional order, called fractional fuzzy optimal control problem (FFOCP). The aim of paper, is to find a fuzzy solution for this kind of problems.

PRELIMINARIES AND NOTATIONS

In this section, first, we recall some necessary definitions and theorems, which are needed for our subject matter about fuzzy and fractional calculus from [10].

Definition 1: Denote E^1 as the set of all functions $x(t)$ that satisfy in the following conditions:

(i) x is normal, i.e. there exist $t \in \mathfrak{R}$, such that $x(t) = 1$.
(ii) x is fuzzy convex, i.e. $\forall\, s, t \in \mathfrak{R}$ and $\lambda \in [0,1]$, $x(\lambda s + (1-\lambda)t) \geq \min\{x(s), x(t)\}$.
(iii) x is upper semi-continuous.
(iv) cl $\{s \in \mathfrak{R} | x(s) > 0\}$, is compact in $\mathfrak{R}$.

Definition 2: The α-level set of a fuzzy number $\in E^1$ where $0 \leq \alpha \leq 1$ is denoted by x_α and is defined as:

$$x_\alpha = \begin{cases} \{s \in \mathfrak{R} | x(s) \geq \alpha\} & 0 < \alpha \leq 1 \\ \mathrm{cl}\,\{s \in \mathfrak{R} | x(s) > 0\} & \alpha = 0 \end{cases} \tag{2}$$

If $x \in E^1$, then x is fuzzy convex, so x_α is closed and bounded in $\mathfrak{R}$, i.e. $x_\alpha \equiv [\underline{x}_\alpha, \overline{x}_\alpha]$, where $\underline{x}_\alpha = \inf\{s \in \mathfrak{R} | x(s) \geq \alpha\} > -\infty$ and $\overline{x}_\alpha = \sup\{s \in \mathfrak{R} | x(s) \geq \alpha\} < \infty$.

Lemma 1: Denote $I = [0,1]$. Assumed that $a: I \to \mathfrak{R}$ and $b: I \to \mathfrak{R}$ satisfy the following conditions:

(i) a and b are bounded non-decreasing functions.
(ii) $a(1) \leq b(1)$.
(iii) For $0 < k \leq 1$, $\lim_{\alpha \to k^-} a(\alpha) = a(k)$ and $\lim_{\alpha \to k^-} b(\alpha) = b(k)$.
(iv) $\lim_{\alpha \to 0^+} a(\alpha) = a(0)$ and $\lim_{\alpha \to 0^+} b(\alpha) = b(0)$.

Then, $\eta: I \to \mathfrak{R}$ defined by $\eta(x) = \sup\{\alpha | a(\alpha) \leq x \leq b(\alpha)\}$ is a fuzzy number with parameterization given by $\{(a(\alpha), b(\alpha), \alpha) | 0 \leq \alpha \leq 1\}$; moreover, if $\hat{\eta}: I \to \mathfrak{R}$ is any fuzzy number with parameterization given by $\{(\hat{a}(\alpha), \hat{b}(\alpha), \alpha) \big| 0 \leq \alpha \leq 1\}$ then functions $\hat{a}(\alpha)$ and $\hat{b}(\alpha)$ satisfy the above conditions (i) _ (iv).

Proof : See [2].

Definition 3: Assume each entry of the vector x be a fuzzy number at the time instant t where [11, 12]:

$$x_\alpha^k = \left[\underline{x}_\alpha{}^k, \overline{x}_\alpha{}^k\right] \qquad k = 1,2, \dots, n. \tag{3}$$

For each $\alpha \in [0,1]$.

Then one can define it as a complex variable like follow [10, 13-15] :

$$x_\alpha^k = \underline{x}_\alpha{}^k + i\overline{x}_\alpha{}^k \qquad k = 1,2, \dots, n. \tag{4}$$

Definition 4: For given $\alpha \in [0,1]$ and arbitrary $\tilde{x} = (\underline{x}_\alpha, \overline{x}_\alpha), \tilde{y} = (\underline{y}_\alpha, \overline{y}_\alpha)$ if k be a real number, we define addition $\tilde{x} + \tilde{y}$, subtraction $\tilde{x} - \tilde{y}$ and scalar multiplication by k as [11, 12]:

$$\tilde{x} + \tilde{y} = \left(\underline{x}_\alpha + \underline{y}_\alpha, \overline{x}_\alpha + \overline{y}_\alpha\right);$$
$$\tilde{x} - \tilde{y} = \left(\underline{x}_\alpha - \underline{y}_\alpha, \overline{x}_\alpha - \overline{y}_\alpha\right);$$
$$k \odot \tilde{x} = \begin{cases} (k\underline{x}_\alpha, k\overline{x}_\alpha), & k \geq 0; \\ (k\overline{x}_\alpha, k\underline{x}_\alpha), & k < 0; \end{cases}$$

Note that we can write any fuzzy number by an interval using parameterization α-level. Assume that $\tilde{x} = (p, q, r)$ be a triangular fuzzy number, we can show this number by α-level parameterization, as follow [10]:

$$\tilde{x} = [q\alpha + p(1 - \alpha), q\alpha + r(1 - \alpha)], \qquad \alpha \in [0,1]. \tag{5}$$

FUZZY RIEMANN-LIOUVILLE DIFFERENTION

Regarding the governing system of our optimal control problem (problem (1)), this section, is devoted to introduce definition of fuzzy Riemann-Liouville integrals and derivatives by Hukuhara difference. Note that $C^F[a, b]$ is the space of all continuous fuzzy-valued function on $[a, b]$ and $L^F[a, b]$ is the space of all Lebesque integrable fuzzy-valued functions on the bounded interval $[a, b] \subset R$.

Definition 5: Let $f \in C^F[a, b] \cap L^F[a, b]$. Then fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following [16, 17]:

$$\left(I_{a+}^{\beta} f\right)(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\beta}} \quad , \ 0 < \beta \leq 1. \tag{6}$$

$$\left(I_{b-}^{\beta} f\right)(x) = \frac{1}{\Gamma(\beta)} \int_x^b \frac{f(t)dt}{(x-t)^{1-\beta}} \quad , \ 0 < \beta \leq 1. \tag{7}$$

$\left(I_{a+}^{\beta} f\right)(x)$ is called the left-sided Riemann-Liouville integral and $\left(I_{b-}^{\beta} f\right)(x)$ is called the right-sided Riemann-Liouville integral of the function $f(.)$ of order β.
Consider the α-cut representation of fuzzy-valued function $f \in C^F[a, b] \cap L^F[a, b]$ as $f(x; \alpha) = \left[\underline{f}(x; \alpha), \overline{f}(x; \alpha)\right]$ for $0 \leq \alpha \leq 1$, where $\underline{f}(x; \alpha)$ and $\overline{f}(x; \alpha)$ are defined as lower bound and upper bound of α-level set of f.

Theorem 1: Let $f \in C^F[a, b] \cap L^F[a, b]$ is a fuzzy-valued function. The fuzzy Riemann-Liouville integral of a fuzzy-valued function f can be expressed as follow [16] :

$$\left(I_{a+}^{\beta} f\right)(x; \alpha) = \left[\left(I_{a+}^{\beta} \underline{f}\right)(x; \alpha), \left(I_{a+}^{\beta} \overline{f}\right)(x; \alpha)\right] \quad , \ 0 \leq \alpha \leq 1 \tag{8}$$

Where

$$\left(I_{a+}^{\beta} \underline{f}\right)(x; \alpha) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\underline{f}(t;\alpha)dt}{(x-t)^{1-\beta}};$$
$$\left(I_{a+}^{\beta} \overline{f}\right)(x; \alpha) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\overline{f}(t;\alpha)dt}{(x-t)^{1-\beta}}.$$

Proof : See [16].

The same formula, can be presented have for $\left(I_{b-}^{\beta} f\right)(x; \alpha)$, as well.
Now, we are going to define the fuzzy Riemann-Liouvelle derivation of order $0 < \beta \leq 1$ for fuzzy-valued function f; for this purpose, we follow in [17].

Definition 6: Let $\in C^F[a,b] \cap L^F[a,b]$, $x_0 \in (a,b) \subseteq \mathcal{R}$ and denote: $\Phi(x) \equiv \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f(t)}{(x-t)^\beta}$. Then f is Riemann-Liouville H-differentiable of order $0 < \beta \leq 1$ at x_0, if there exist an element $\left(D_{a+}^\beta f\right)(x_0) \in E$, such that for $h > 0$ sufficiently small:

$$\left(D_{a+}^\beta f\right)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0 + h) \ominus \Phi(x_0)}{h} = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 - h)}{h}$$

Theorem 2: let $f \in C^F[a,b] \cap L^F[a,b]$, $x_0 \in (a,b)$ and $0 < \beta \leq 1$, then:

$$\left(D_{a+}^\beta f\right)(x;\alpha) = \left[\left(D_{a+}^\beta \underline{f}\right)(x;\alpha), \left(D_{a+}^\beta \overline{f}\right)(x;\alpha)\right] \ , \ 0 \leq \alpha \leq 1$$

Similarly, we have the results for $\left(D_{b-}^\beta f\right)(x;\alpha)$.
Proof : See [16].

Now, we want to discuss about fractional Pontryagin's systems. Consider the fractional optimal control problem as following :

$$Min: \int_a^b f_0(t, x(t), u(t))dt \tag{9}$$

S. to :

$$\begin{cases} \left(D_{b-}^\beta x\right)(t) = f\left(t, x(t), u(t)\right) \\ \qquad x(a) = A, \end{cases}$$

that A is a real number. A necessary condition for (x^*, u^*) to be a solution of (9) is that there exist a function w such that the following fractional Pontryagin's system holds [18]:

$$\begin{cases} D_{b-}^\beta x = \dfrac{\partial H}{\partial w}(x,u,w,t); \\ D_{a+}^\beta w = \dfrac{\partial H}{\partial x}(x,u,w,t); \\ \dfrac{\partial H}{\partial u}(x,u,w,t) = 0; \\ \left(x(a), w(b)\right) = (A, 0); \end{cases}$$

where $H(x,u,w,t) = f_0(t,x,u) + w.f(x,u,t)$. We refer to [18].

FRACTIONAL FUZZY OPTIMAL CONTROL PROBLEM

Based on the above discussion, we are going to present a solution method for FFOCPs. Consider the following fractional fuzzy optimal control problem :

$$Min \int_a^b f_0(t, \tilde{x}(t), \tilde{u}(t))dt \tag{10}$$

S. to :

$$\begin{cases} \left(D_{a+}^\beta \tilde{x}\right)(t) = f\left(t, \tilde{x}(t), \tilde{u}(t)\right); \\ \tilde{x}(a) = \tilde{x}_0 = (p,q,r), \end{cases}$$

That in initial condition, $\tilde{x}_0 = (p,q,r)$ is a triangular fuzzy number and $0 < \beta \leq 1$.
By using the concept of α-cut, theorem 2 and parameterization of a fuzzy number, we can write problem (10) in complex space as follows for each $0 \leq \alpha \leq 1$:

$$Min: \int_a^b f_0\left(t, \underline{x}(t;\alpha), \underline{u}(t;\alpha)\right) + i f_0\big(t, \overline{x}(t;\alpha), \overline{u}(t;\alpha)\big) dt \tag{11}$$

S.to :

$$\begin{cases} \big(D_{a+}^{\beta}\underline{x}\big)(t;\alpha) + i\big(D_{a+}^{\beta}\overline{x}\big)(t;\alpha) = f\left(t, \underline{x}(t;\alpha), \underline{u}(t;\alpha)\right) + i f\big(t, \overline{x}(t;\alpha), \overline{u}(t;\alpha)\big); \\ \underline{x}(a;\alpha) + i\overline{x}(a;\alpha) = \big(q\alpha + p(1-\alpha)\big) + i\big(q\alpha + r(1-\alpha)\big); \end{cases}$$

The new representation of the problem can be turned in to two problems (12) and (13):

$$Min: \int_a^b f_0\left(t, \underline{x}(t;\alpha), \underline{u}(t;\alpha)\right) dt \tag{12}$$

S. to :

$$\begin{cases} \big(D_{a+}^{\beta}\underline{x}\big)(t;\alpha) = f\left(t, \underline{x}(t;\alpha), \underline{u}(t;\alpha)\right); \\ \underline{x}(a;\alpha) + i\overline{x}(a;\alpha) = \big(q\alpha + p(1-\alpha)\big). \end{cases}$$

$$Min: \int_a^b f_0\big(t, \overline{x}(t;\alpha), \overline{u}(t;\alpha)\big) dt \tag{13}$$

S. to :

$$\begin{cases} \big(D_{a+}^{\beta}\overline{x}\big)(t;\alpha) = f\big(t, \overline{x}(t;\alpha), \overline{u}(t;\alpha)\big); \\ \overline{x}(a;\alpha) = \big(q\alpha + r(1-\alpha)\big). \end{cases}$$

By solving this two problems we obtain the optimal pairs $(\underline{x}^*(t;\alpha), \underline{u}^*(t;\alpha))$ and $(\overline{x}^*(t;\alpha), \overline{u}^*(t;\alpha))$ respectively for (12) and (13) for any given α. Therefore, a solution for (10) can be constructed as :

$$\tilde{x}^*(t,\alpha) = [\,\underline{x}^*(t;\alpha), \overline{x}^*(t;\alpha)];$$
$$\tilde{u}^*(t,\alpha) = \big[\,\underline{u}^*(t;\alpha), \overline{u}^*(t;\alpha)\big].$$

Note 1: Consider the Cauchy type problem [16] :

$$\begin{cases} (D_{a+}^{\beta}y)(x) = f(x, y(x)) & 0 < \beta \le 1 \\ (I_{a+}^{1-\beta}y)(a) = b, & b \in R \end{cases} \tag{14}$$

that $f(x, y(x))$ is a real-valued continuous function in domain $G \subset R \times R$ such that $sup_{(x,y)\in G}|f(x,y)| \le \infty$ and that is satisfies the Lipschitz condition. The solution of this fractional system is also given in [16] as :

$$y(x) = \frac{b(x-a)^{\beta-1}}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t, y(t))dt}{(x-t)^{1-\beta}}\,, \qquad x > a\,, 0 < \beta \le 1. \tag{15}$$

Also, consider the following Cauchy type problem for linear differential equation [16] :

$$\begin{cases} (D_{a+}^{\beta}y)(x) - \lambda y(x) = f(x); \\ y(a) = b; \end{cases} \tag{16}$$

Then the solution of this system is :

$$y(x) = b x^{\beta-1} E_{\beta,\beta}\big(\lambda(x-a)^{\beta}\big) + \int_a^x (x-t)^{\beta-1} E_{\beta,\beta}\big(\lambda(x-t)^{\beta}\big) f(t) dt, \tag{17}$$

where $E_{\beta,\beta}$ is Mittag-Leffler function, that in general form the Mittage-Leffler function is defined by [16]:

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad z \in \mathbb{C};\ \Re(\alpha) > 0. \tag{18}$$

(For more details, see [16]).

Theorem 3: Let $0 < \beta \leq 1$ and $0 < a < b < \infty$.
(a) The equality $(D_{a+}^{\beta} y)(x) = 0$ is valid if, and only if, $y(x) = c(log\frac{x}{a})^{\beta-1}$ where $c \in R$.
(b) The equality $(D_{b-}^{\beta} y)(x) = 0$ is valid if, and only if, $y(x) = d(log\frac{b}{x})^{\beta-1}$ where $d \in R$.

Proof : see [16].

EXAMPLE

Consider the fractional fuzzy optimal control problem :

$$Min: \int_0^1 \left(\frac{\tilde{u}(t)^2}{2} + q(1-t)^p \tilde{x}(t)\right) dt \tag{19}$$

S. to :

$$\begin{cases} \left(D_{0^+}^{\beta} \tilde{x}\right)(t) = \lambda \tilde{x}(t) + \mu \tilde{u}(t); \\ \tilde{x}(0) = (0,1,2), \end{cases}$$

where $\tilde{x}(t)$ is fuzzy variable, $\tilde{u}(t)$ is fuzzy control variable, $t \in [0,1]$, $0 < \beta \leq 1$ and $p, q, \lambda, \mu \in R^+$. Based on (11), first we represent the α-cut for the problem in complex space:

$$Min: \int_0^1 \left(\frac{1}{2}(\underline{u}(t;\alpha)^2 + i\overline{u}(t;\alpha)^2) + q(1-t)^p(\underline{x}(t;\alpha) + i\overline{x}(t;\alpha))\right) dt \tag{20}$$

S. to :

$$\begin{cases} \left(D_{0^+}^{\beta} \underline{x}\right)(t;\alpha) + i\left(D_{0^+}^{\beta} \overline{x}\right)(t;\alpha) = \lambda\left(\underline{x}(t;\alpha) + i\overline{x}(t;\alpha)\right) + \mu\left(\underline{u}(t;\alpha) + i\overline{u}(t;\alpha)\right); \\ \underline{x}(0;\alpha) + i\overline{x}(0;\alpha) = \left(\alpha + 0(1-\alpha)\right) + i\left(\alpha + 2(1-\alpha)\right). \end{cases}$$

According to the problems (12) and (13), we divide (20) in to the following problems :

$$Min \int_0^1 \left(\frac{1}{2}\underline{u}(t;\alpha)^2 + q(1-t)^p \underline{x}(t;\alpha)\right) dt \tag{21}$$

S. to :

$$\begin{cases} \left(D_{0^+}^{\beta} \underline{x}\right)(t;\alpha) = \lambda \underline{x}(t;\alpha) + \mu \underline{u}(t;\alpha); \\ \underline{x}(0;\alpha) = \alpha. \end{cases}$$

$$Min \int_0^1 \left(\frac{1}{2}\overline{u}(t;\alpha)^2 + q(1-t)^p \overline{x}(t;\alpha)\right) dt \tag{22}$$

S. to :

$$\begin{cases} \left(D_{0^+}^{\beta} \overline{x}\right)(t;\alpha) = \lambda \overline{x}(t;\alpha) + \mu \overline{u}(t;\alpha); \\ \overline{x}(0;\alpha) = \alpha + 2(1-\alpha). \end{cases}$$

Now, we try to solve problem (21), by defining the Hamiltonian function :

$$H\left(\underline{x}(t;\alpha), \underline{u}(t;\alpha), w, t\right) = \left(\frac{1}{2}\underline{u}(t;\alpha)^2 + q(1-t)^p \underline{x}(t;\alpha)\right) + w.(\lambda \underline{x}(t;\alpha) + \mu \underline{u}(t;\alpha))$$

And using fractional Pontryagin system, as fallowing :

$$\begin{cases} D_{0^+}^{\beta}\,\underline{x}(t;\alpha) = \dfrac{\partial H}{\partial w}\big(\underline{x}(t;\alpha),\underline{u}(t;\alpha),w,t\big) = \lambda\underline{x}(t;\alpha) + \mu\underline{u}(t;\alpha); & (23) \\ D_{0^+}^{\beta}\,w = \dfrac{\partial H}{\partial \underline{x}(t;\alpha)}\big(\underline{x}(t;\alpha),\underline{u}(t;\alpha),w,t\big) = q(1-t)^p + \lambda w; & (24) \\ \dfrac{\partial H}{\partial u}(x,u,w,t) = \underline{u}(t;\alpha) + \mu w = 0; & (25) \\ \Big(\underline{x}(0;\alpha), w(1)\Big) = (\alpha, 0); & (26) \end{cases}$$

From (24) and (25) together, we have :

$$\begin{cases} D_{0^+}^{\beta}\,w = q(1-t)^p + \lambda w; \\ w(1) = 0, \end{cases} \tag{27}$$

where $t \in [0,1]$, that fractional equation (27) is equivalent to:

$$\begin{cases} D_{1^-}^{\beta}\,w' = qt^p + \lambda w'; \\ w'(0) = 0, \end{cases} \tag{28}$$

where $w'(t) = w(1-t)$. The unique solution of the fractional Cauchy problem (28) can be given from [16, chap. 3] as :

$$w'(t) = \int_0^t (t-y)^{\beta-1} E_{\beta,\beta}\big(\lambda(t-y)^{\beta}\big) q y^p \, dy \quad , \qquad \forall\, t \in [0,1] \tag{29}$$

where $E_{\beta,\beta}$ is Mittag-Leffler function defined by (18). Now, we make a change of variable to have a better formula :

$$\begin{aligned} w'(t) &= q\int_0^t (t-y)^p y^{\beta-1} E_{\beta,\beta}\big(\lambda y^{\beta}\big)\, dy = q\Gamma(p+1) I_{1^-}^{p+1}\Big(y^{\beta-1}E_{\beta,\beta}\big(\lambda y^{\beta}\big)\Big)(t) \\ &= q\Gamma(p+1)t^{\beta+p}E_{\beta,\beta+p+1}\big(\lambda t^{\beta}\big). \end{aligned} \tag{30}$$

For more details, refer to [16]. So, we have :

$$w(t) = w'(1-t) = q\Gamma(p+1)(1-t)^{\beta+p}E_{\beta,\beta+p+1}\big(\lambda(1-t)^{\beta}\big). \tag{31}$$

Now from (25) and (31) we have :

$$\underline{u}^*(t;\alpha) = q\Gamma(p+1)(1-t)^{\beta+p}E_{\beta,\beta+p+1}\big(\lambda(1-t)^{\beta}\big), \tag{32}$$

Substituting (32) in (23) :

$$\begin{cases} D_{0^+}^{\beta}\,\underline{x}(t;\alpha) = \lambda\underline{x}(t;\alpha) + \mu\big[q\Gamma(p+1)(1-t)^{\beta+p}E_{\beta,\beta+p+1}\big(\lambda(1-t)^{\beta}\big)\big]; \\ \underline{x}(0;\alpha) = \alpha. \end{cases} \tag{33}$$

Then, from (17), one can introduce the optimal trajectory as :

$$\underline{x}^*(t;\alpha) = \int_0^t (t-y)^{\beta-1} E_{\beta,\beta}\big(\lambda(t-y)^{\beta}\big)\big[\mu q\Gamma(p+1)(1-y)^{\beta+p}E_{\beta,\beta+p+1}\big(\lambda(1-y)^{\beta}\big)\big]dy + \alpha t^{\beta-1}E_{\beta,\beta}\big(\lambda t^{\beta}\big) \tag{34}$$

Therefore, we obtain the optimal pair $(\underline{x}^*(t;\alpha), \underline{u}^*(t;\alpha))$ the solution of (21). By solving problem (22) (i.e. $\Big(\overline{x}^*(t;\alpha), \overline{u}^*(t;\alpha)\Big)$) in similar way, can be determined. Thus the optimal solution of (19) is :

$$\overline{u}^*(t;\alpha) = q\Gamma(p+1)(1-t)^{\beta+p}\mathrm{E}_{\beta,\beta+p+1}\big(\lambda(1-t)^{\beta}\big) \tag{35}$$

And

$$\overline{x}^*(t;\alpha) = (\alpha + 2(1-\alpha))t^{\beta-1}E_{\beta,\beta}(\lambda t^\beta) + \int_0^t (t-y)^{\beta-1} E_{\beta,\beta}(\lambda(t-y)^\beta)[\mu q\Gamma(p+1)(1-y)^{\beta+p}E_{\beta,\beta+p+1}(\lambda(1-y)^\beta)]dy \quad (36)$$

as the optimal fuzzy solution of fractional fuzzy control problem (19), for given $\alpha, \beta, \lambda, \mu, p$ *and q*, where $0 \le \alpha \le 1$, $0 < \beta \le 1$ and $p, q, \lambda, \mu \in R^+$.

CONCLUSION

In this paper, by introducing the fractional-fuzzy optimal control problems, a method for solving a class of these problems by using Mittag-Leffler function is presented. When the decision maker specifies a value for α (0≤α≤1). Also, an example is presented to apply this method that one can use the presented method to obtain the optimal trajectory and optimal control for the fractional fuzzy optimal control problems.

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